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On finitistic spaces

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Abstract

Finitistic spaces form a natural class containing compact and finite-dimensional spaces. Introduced and investigated by fixed-point theorists, finitistic spaces found an application in cohomological dimension theory. In the paper, two characterizations of paracompact, finitistic spaces are proved. These characterizations allow to create a mechanism of generalizing results of finite dimension theory. As an application we obtain results on compact group actions on paracompact spaces which were previously known for compact Lie group actions. © 1999 Published by Elsevier Science B.V.

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0. Introduction

The classical cohomological methods in the study of group actions were applied either to compact Hausdorff spaces or paracompact spaces of finite cohomological dimension (see [1,2]). Swan [27] introduced the concept of finitistic spaces and obtained results generalizing classical Smith-type fixed point theorems.

Definition of finitistic spaces 0.1. A space X is *finitistic* if every open cover of X has an open refinement of finite order.

Recall that the order of an open cover is at most n if every $n+2$ distinct elements of the cover have empty intersection. Since finite open covers have finite order, compact

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Hausdorff spaces are finitistic. Also paracompact spaces of finite covering dimension are finitistic. Indeed, $\dim(X) \leqslant n$, where $0 \leqslant n < \infty$, if every finite open cover of X has a finite open refinement of order at most n, and in the case of paracompact space X one can find open refinement of order at most n for every open cover of X (see [20, Lemma 3.1.9, p. 172]). Thus, the class of paracompact, finitistic spaces may be formally viewed as a natural class combining both compact and finite-dimensional spaces. One of the goals of this paper is to show that the connection of paracompact, finitistic spaces to compact/finite-dimensional spaces is much closer than previously thought (see Theorem 0.5).

After Swan's introduction of finitistic spaces, Bredon [2] set the trend of stating results on the cohomological structure of fixed point sets in terms of finitistic spaces and it is apparent that finitistic spaces give a natural environment for using the Čech method in generalizing cohomological results on fixed point sets and orbit spaces. As seen in [5], a typical result on toral actions and rational coefficients involves the assumption that both the total space X and the orbit space X/T^n (T^n being the n-torus) are finitistic. In an effort to weaken those assumptions, Deo and Tripathi proved the following:

Theorem 0.2 [6]. *Suppose a compact Lie group G acts on a paracompact, finitistic space X. Then, the orbit space X/G is finitistic.*

The converse of that statement was proved by Deo and Singh:

Theorem 0.3 [4]. *Suppose a compact Lie group G acts on a paracompact X. If the orbit space X/G is finitistic, then X is finitistic, too.*

Another branch of topology where finitistic spaces surfaced recently is the cohomological dimension theory. Namely, Rubin and Schapiro obtained the following:

Theorem 0.4 [25]. *Suppose X is paracompact, finitistic, and G is finitely generated, Abelian group. Then*

(a) $\dim_G(X) = \dim_G(\beta X)$, *where βX is the Čech–Stone compactification of X,*
(b) *if X is metric, separable, finitistic spaces, then X has a metric compactification preserving cohomological dimension.*

Earlier, Dranishnikov [9] found a metric, separable space X so that $\dim_{\mathbb{Z}}(X) = 4$ and $\dim_{\mathbb{Z}}(\beta X) > 4$, and Dydak and Walsh [15] found a metric, separable space X so that $\dim_{\mathbb{Z}}(X) = 4$ and $\dim_{\mathbb{Z}}(\kappa X) > 4$ for any compactification κX of X.

The purpose of this paper is to develop characterizations of finitistic spaces and use these to generalize above mentioned results of [6,4,25]. Basically speaking, our characterizations enable us to create a mechanism of generalizing results of finite dimension theory to the realm of finitistic spaces. Here is the most important characterization:

Theorem 0.5. *A paracompact space X is finitistic iff there is a compact subset Z of X so that $X - U$ is finite-dimensional for every open neighborhood U of Z in X.*

Its predecessor is a characterization due to Deo and Tripathi:

Theorem 0.6 [6]. *A paracompact space X is not finitistic iff there is a discrete family of closed subsets $\{A_n\}_{n\geqslant 1}$ of X such that* $\dim(A_n) \geqslant n$ *for each n.*

It is easy to deduce Theorem 0.6 from Theorem 0.5 and the latter establishes a much closer connection of paracompact, finitistic spaces to compact/finite-dimensional spaces. Theorem 0.5 is our main vehicle in generalizing results of finite dimension theory to the realm of finitistic spaces.

1. K-approximations

It is well known that the covering dimension of normal spaces can be characterized in terms of extension of maps into spheres (see [20, 3.2.10, p. 188], or [22]). There is a dual characterization in terms of approximating maps by maps into n-dimensional polyhedra. Namely, it is well known that X is at most n-dimensional iff any map from X to a metric simplicial complex L (i.e., a simplicial complex equipped with the metric topology) can be approximated by a map to the n-skeleton $L^{(n)}$ of L. This characterization was extended in [18] to the case of cohomological dimension with respect to some groups. We would like to characterize finitistic spaces in a similar vein, and it is achieved with the notion of a K-approximation. One may view this part of the paper is an implementation of ideas from [13] where it is shown that several concepts/results of set-theoretic topology can be presented more conveniently by using partitions of unity rather than open covers. In this section we concentrate on maps to metric simplicial complexes which, as seen in [13], correspond to point-finite partitions of unity.

Definition of K-approximations 1.1. Let K be a metric simplicial complex. A map $g : X \to K$ is a *K-approximation* of $f : X \to K$ provided for each simplex Δ of K and each $x \in X$, $f(x) \in \Delta$ implies $g(x) \in \Delta$. g is an *n-dimensional* (respectively, *finite-dimensional*) *K-approximation* of f if it is a K-approximation and $g(X) \subset K^{(n)}$ (respectively, $g(X) \subset K^{(m)}$ for some m).

Notice that for each $x \in X$ there is an unique simplex Δ_x of K so that the geometric interior $\mathrm{Int}(\Delta_x)$ of Δ_x contains $f(x)$. Also, notice that g is a K-approximation of f iff $g(x) \in \Delta_x$ for all $x \in X$.

Basic Observations 1.2. *Suppose $f : X \to K$ is a map and K is a metric simplicial complex. Let $S = K^{(0)}$ be the set of vertices of K which is considered as a subset of l^1_S, the Banach space of all absolutely summable S-sequences $\{a_s\}_{s\in S}$.*

(a) *The set of all K-approximations of f is a convex subset of the vector space $(l^1_S)^X$ of maps from X to l^1_S.*

(b) *The convex hull of the set of all n-dimensional K-approximations of f is contained in the set of all $(2n+1)$-dimensional K-approximations of f.*

(c) *The set of all finite-dimensional K-approximations of f is convex.*
(d) *If g is a K-approximation of f and h is a K-approximation of g, then h a K-approximation of f.*

Proof. Suppose $g_1, g_2 : X \to K$ are two K-approximations of f, and $t \in [0,1]$. Let $g = t \cdot g_1 + (1-t) \cdot g_2$. Notice that if $f(x) \in \Delta$, then $g_1(x), g_2(x) \in \Delta$ which implies $g(x) \in \Delta$. Thus, (a) holds. Moreover, if both $g_1(x)$ and $g_2(x)$ belong to the n-skeleton of Δ, then $g(x)$ belongs to the $(2n+1)$-skeleton of Δ. This proves (b), and (b) implies (c). The proof of (d) is left is an easy exercise. □

An immediate application of Basic Observations 1.2 is:

Corollary 1.3. *Every two K-approximations of f are homotopic.*

Our next result is a more sophisticated version of Basic Observations 1.2:

Lemma 1.4. *Let $f : X \to K$ be a map from a normal space X to a metric simplicial complex K. Suppose $g_i : U_i \to K$ is a K-approximation of $f|U_i$, $i = 1, 2$, for some open subsets U_i of X and $X = U_1 \cup U_2$. There is a K-approximation $g : X \to K$ of f so that $g|(U_i - U_j) = g_i|(U_i - U_j)$ for all $i, j \leqslant 2$. Moreover, if each g_i is finite-dimensional, then so is g.*

Proof. Choose a map $\gamma : X \to [0,1]$ so that $\gamma|(X - U_1) = 0$ and $\gamma|(X - U_2) = 1$. Define $g : X \to K$ by $h(x) = \gamma(x) \cdot g_1(x) + (1 - \gamma(x)) \cdot g_2(x)$ if $x \in U_1 \cap U_2$, $g(x) = g_1(x)$ if $x \in U_1 - U_2$, and $g(x) = g_2(x)$ if $x \in U_2 - U_1$. □

Result Lemma 1.4 should be thought of as a pasting of two local K-approximations of f. As an application we can extend a local K-approximation to a global K-approximation:

Corollary 1.5. *Let $f : X \to K$ be a map from a normal space X to a metric simplicial complex K. Suppose $g : U \to K$ is a K-approximation of $f|U$ for an open neighborhood U of a closed subset A of X. There is a K-approximation $h : X \to K$ of f with $h|A = g|A$.*

Proof. Choose an open neighborhood V of A in U with $\mathrm{cl}_X(V) \subset U$. Put $U_1 = U$, $U_2 = X - \mathrm{cl}_X(V)$, $g_1 = g$, and $g_2 = f|U_2$. Apply Lemma 1.4. □

Our next result is the first step in constructing local n-dimensional approximations:

Lemma 1.6. *Suppose K is a metric simplicial complex and $n \geqslant 0$. There is a K-approximation h of the identity map $id : K \to K$ so that $h|K^{(n)} = id|K^{(n)}$ and $h|U$ is n-dimensional for some neighborhood U of $K^{(n)}$.*

Proof. It is well known that there is a neighborhood V of $C = K^{(n)}$ in X and a retraction $r : V \to C$. Typically (see [24, Lemma 6, p. 305]), one takes the first barycentric subdivision K' of K, V is the union of geometric interiors of all simplices in K' which intersect C, and if $x = \sum_{v\in(K')^{(0)}} c_v \cdot v$, then

$$r(x) = \Bigl(\sum_{v\in(K')^{(0)}\cap C} c_v \cdot v\Bigr) \Big/ \Bigl(\sum_{v\in(K')^{(0)}\cap C} c_v\Bigr).$$

Notice that r is an n-dimensional K-approximation of the inclusion $V \hookrightarrow K$. Indeed, if $x \in \Delta$ and Δ is a simplex in K, then $r(x)$ belongs to $C \cap \Delta'$. Choose a closed neighborhood A of C in K with $A \subset V$. By Corollary 1.5, there is a K-approximation of f which is an extension of $r|A$. Put $U = \mathrm{Int}(A)$. $\square$

Corollary 1.7. *Let $f : X \to K$ be a map from a space X to a metric simplicial complex K so that $f(A) \subset K^{(n)}$ for some subset A of X. There is a K-approximation g of f so that $g|U$ is an n-dimensional K-approximation of $f|U$ for some open neighborhood U of A and $g|A = f|A$.*

Proof. Let $h : K \to K$ be as in Lemma 1.6, i.e., h is a K-approximation of identity, $h|K^{(n)} = id|K^{(n)}$, and $h|V$ is n-dimensional for some neighborhood V of $K^{(n)}$. Put $g = h \circ f$ and $U = f^{-1}(V)$. $\square$

Now, we are ready to construct local n-dimensional K-approximations:

Lemma 1.8. *Let $f : X \to K$ be a map from a paracompact space X to a metric simplicial complex K. Suppose there is an n-dimensional K-approximation of $f|A$ for some closed subset A of X. There is an open neighborhood U of A in X and a K-approximation $g : X \to K$ of f so that $g|U$ is an n-dimensional K-approximation of $f|U$.*

Proof. Let $h : A \to K$ be an n-dimensional K-approximation of $f|A$. Since $K^{(n)}$ is an absolute neighborhood extensor of X (see [21, Theorem 11.7, p. 109]), there is an extension $H : V \to K^{(n)}$ of g over a neighborhood V of A. However, H may not be a K-approximation of $f|V$. Let S be the set of vertices of K. Consider the family

$$\mathcal{V} = \bigl\{H^{-1}\bigl(St(v, K)\bigr) \cap f^{-1}\bigl(St(v, K)\bigr)\bigr\}_{v\in S},$$

where $St(v, K)$ is the star of a vertex v of K, i.e., the union of geometric interiors of all simplices of K containing v as a vertex. Since

$$h^{-1}\bigl(St(v, K)\bigr) \subset H^{-1}\bigl(St(v, K)\bigr) \cap f^{-1}\bigl(St(v, K)\bigr) \quad \text{for each } v \in S,$$

$\mathcal{V}$ covers A and its order is at most n. Since X is paracompact, there is a closed neighborhood B of A in X, $B \subset \bigcup \mathcal{V}$, and a partition of unity $\{g_v\}_{v\in S}$ on B so that

$$g_v^{-1}(0, 1] \subset H^{-1}\bigl(St(v, K)\bigr) \cap f^{-1}\bigl(St(v, K)\bigr) \quad \text{for each } v \in S.$$

Notice that $g : B \to K$ defined by $g(x) = \sum_{v\in S} g_v(x) \cdot v$ is an n-dimensional K-approximation of $f|B$. Apply Corollary 1.5. $\square$

Corollary 1.9. *Let $f:X\to K$ be a map from a paracompact space X to a metric simplicial complex K. Consider the set $\mathcal{F}$ of all closed subsets A of X such that $f|A$ has a finite-dimensional K-approximation. $\mathcal{F}$ is closed under finite unions.*

Proof. Suppose $A_i\in\mathcal{F}$ for $i=1,2$. By Lemma 1.8, there are neighborhood U_i of A_i and K-approximations g_i of f so that $g_i|U_i$ is finite-dimensional. By Lemma 1.4 there is a finite-dimensional approximation $h:U_1\cup U_2\to K$ of $f|(U_1\cup U_2)$. By Corollary 1.5, $A_1\cup A_2\in\mathcal{F}$. □

2. Characterizations of finitistic spaces

Now, we can generalize the property of approximating maps of n-dimensional spaces by maps into the n-skeleton in terms of K-approximations of maps on finitistic spaces:

Theorem 2.1. *For a paracompact space X the following conditions are equivalent*:

(a) *X is finitistic,*

(b) *for any metric simplicial complex K every map $f:X\to K$ has a finite-dimensional K-approximation g,*

(c) *for any metric simplicial complex K and every $m\geqslant -1$, every map $f:X\to K$ has a finite-dimensional K-approximation g so that $g|f^{-1}(K^{(m)})=f|f^{-1}(K^{(m)})$.*

Proof. (a) ⇒ (b) Let $f:X\to K$ and let $St(v,K)$ be the star of a vertex v of K, i.e., the union of geometric interiors of all simplices of K containing v as a vertex. Let $S=K^{(0)}$ be the set of vertices of K. Notice that $\{f^{-1}(St(v,K))\}_{v\in S}$ is an open cover of X. Choose an open cover $\{W_t\}_{t\in T}$ of finite order and refining $\{f^{-1}(St(v,K))\}_{v\in S}$. Let $\gamma:T\to S$ be a function so that $W_t\subset f^{-1}(St(\gamma(v),K))$ for all $t\in T$. Define U_v as $\bigcup\{W_t\mid t\in\gamma^{-1}(v)\}$ and notice that $\{U_v\}_{v\in S}$ is an open cover of X of finite order so that $U_v\subset f^{-1}(St(v,K))$ for all $v\in S$. Choose a partition of unity $\{\alpha_v\}_{v\in S}$ of X with $\alpha_v^{-1}(0,1]\subset U_v$ for all $v\in S$ and notice that $g(x)=\sum_{v\in S}\alpha_v(x)\cdot v$ defines a map $g:X\to K$ which is a finite-dimensional K-approximation of f.

(b) ⇒ (c) Let $h:X\to K$ be a finite-dimensional K-approximation of f. By Corollary 1.7 there is a K-approximation $r:X\to K$ of f so that of some neighborhood U_1 of $A=f^{-1}(K^{(m)})$, $r|U_1$ is an m-dimensional K-approximation of $f|U$, and $r|A=f|A$. Choose an open neighborhood U_2 of $X-U_1$ in $X-A$. Apply Lemma 1.4 to produce a finite-dimensional K-approximation g of f with $f|A=g|A$.

(b) is a special case of (c) $(m=-1)$.

(b) ⇒ (a) Suppose $\{U_v\}_{v\in S}$ is an open cover of X. Choose a partition of unity $\{\alpha_v\}_{v\in S}$ of X with $\alpha_v^{-1}(0,1]\subset U_v$ for all $v\in S$ and notice that $f(x)=\sum_{v\in S}\alpha_v(x)\cdot v$ defines a map $f:X\to K$, where K is the full complex with S as its set of vertices. Let g be a finite-dimensional K-approximation of f. Notice that $g^{-1}(St(v,K))\subset U_v$ for all $v\in S$ and $\{g^{-1}(St(v,K))\}_{v\in S}$ is of finite order. □

It is worth proving the analog of Theorem 2.1 for finite-dimensional spaces. The following result was previously known for metric spaces but, as far as the authors know, is new in the case of general paracompact spaces:

Theorem 2.2. *Let n be an integer. For a paracompact space X the following conditions are equivalent*:

(a) $\dim(X) \leqslant n$,

(b) *for any metric simplicial complex K every map $f : X \to K$ has a n-dimensional K-approximation g*,

(c) *for any metric simplicial complex K, every map $f : X \to K$ has a n-dimensional K-approximation g so that $g|f^{-1}(K^{(n)}) = f|f^{-1}(K^{(n)})$.*

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (a), and (c) $\Rightarrow$ (b). An obvious modification of the proof of Theorem 2.1 works in this case. Only (b) $\Rightarrow$ (c) requires proof. Suppose $f : X \to K$ is a map. Let $h : X \to K$ be an m-dimensional K-approximation of f so that $h|f^{-1}(K^{(n)}) = f|f^{-1}(K^{(n)})$. Such an approximation exists as X is finitistic (see Theorem 2.1). We may assume that m is the smallest integer so that such an approximation h exists. We need to prove $m \leqslant n$. Suppose, on the contrary, that $m > n$. Notice that barycenters of all m-dimensional simplices in $K^{(m)}$ form a discrete subset of $K^{(m)}$. Therefore, for each m-simplex Δ of $K^{(m)}$ we may find an m-ball B_Δ in the geometric interior of Δ so that the collection of those balls is discrete in $K^{(m)}$. Let $A = h^{-1}(\mathrm{Bd}(B_\Delta))$. Since $\mathrm{Bd}(B_\Delta)$ is an $(m-1)$-sphere, it is an absolute extensor of $h^{-1}(B_\Delta)$ and one can modify h on $h^{-1}(B_\Delta)$ so that the image of the new map misses the interior of B_Δ. This is done by extending $h|A : A \to \mathrm{Bd}(B_\Delta)$ over $h^{-1}(B_\Delta)$. Denote the modified map by h'. Now, for each m-simplex Δ of K there is a retraction $r_\Delta : \Delta - \mathrm{Int}(B_\Delta) \to \partial\Delta$ and all these retractions can be pasted together to give a retraction $r : h'(X) \cup K^{(m-1)} \to K^{(m-1)}$. Notice that $r \circ h'$ is an $(m-1)$-dimensional approximation of f contradicting the minimality of m. Thus, $m \leqslant n$. $\square$

With the help of Theorem 2.1 we are now able to prove a generalization of part of Theorem 0.5:

Corollary 2.3. *Suppose A is finitistic and is a closed subset of a paracompact space X. If $X - U$ is finitistic for every open neighborhood U of A in X, then X is finitistic.*

Proof. Suppose $f : X \to K$ is a map and K is a metric simplicial complex. Consider the set $\mathcal{F}$ of all closed subsets B of X such that $f|B$ has a finite-dimensional K-approximation. Since A is finitistic, $A \in \mathcal{F}$ and, by Lemma 1.8 there is an open neighborhood V of A in X such that $\mathrm{cl}(V) \in \mathcal{F}$. Choose an open neighborhood U of A in V so that $\mathrm{cl}_X(U) \subset V$. Now, $X - U$ is finitistic, so $X - U \in \mathcal{F}$. Also, $\mathrm{cl}_X(U) \in \mathcal{F}$ which, by Corollary 1.9, implies that $X \in \mathcal{F}$ as $X = (X - U) \cup \mathrm{cl}_X(U)$. $\square$

We are now ready to prove Theorem 0.5 in full generality:

Structure Theorem 2.4. *A paracompact space X is finitistic iff there is a compact subset Z of X so that $X - U$ is finite-dimensional for every open neighborhood U of Z in X.*

Proof. Part of Theorem 2.4 follows from Corollary 2.3. Assume X is finitistic and let Z be the set of all points $x \in X$ so that every closed neighborhood A of x is infinite-dimensional. Clearly, Z is a closed subset of X. To prove that Z is compact it suffices to show it does not have a countable discrete subset (see [19]). So, suppose $B = \{x_n\}_{n \geqslant 1}$ is a discrete subset of Z. By extending the map $f : B \to R$, $f(x_n) = n$ for each $n \geqslant 1$, to $F : X \to R$ we can put $B_n = F^{-1}[n - 1/3, n + 1/3]$ and consider C as the union of all B_n, $n \geqslant 1$. Since each B_n is infinite-dimensional, there is an open cover $\mathcal{U}_n$ of B_n with no refinement of order at most n. Now, $\mathcal{U} = \bigcup \mathcal{U}_n$ is an open cover of C with no refinement of finite order, contradicting C being finitistic.

Now, suppose $X - U$ is not finite-dimensional for some open neighborhood of Z. Since $X - U$ is locally finite-dimensional, there is a sequence $\{x_n\}_{n \geqslant 1}$ of points in $X - U$ such that every closed neighborhood A of x_n in $X - U$ is at least n-dimensional. As above, $B = \{x_n\}_{n \geqslant 1}$ cannot be discrete. However, any point $y \in \mathrm{cl}_X(B) - B$ must belong to Z contradicting $\mathrm{cl}_X(B) \subset X - U$. □

Corollary 2.5. *Suppose X is a paracompact space such that $X = A_1 \cup A_2$. If both A_1 and A_2 are paracompact and finitistic, then X is finitistic if one of the following conditions are satisfied*:

(a) *A_1 is closed in X,*
(b) *X is metrizable.*

Proof. If A_1 is closed, then for every open neighborhood U of A_1 in X, $X - U = A_2 - U$ is a closed subset of A_2 and is finitistic. Apply Corollary 2.3.

Suppose X is metrizable. Choose compact subsets Z_i of A_i so that, for every open neighborhood U of Z_i in A_i, $A_i - U$ is finite-dimensional. Suppose U is an open neighborhood of $Z_1 \cup Z_2$. Since both $A_i - U$, $i = 1, 2$, are finite-dimensional, so is $X - U = (A_1 - U) \cup (A_2 - U)$ (see [20, 3.1.17 on p. 174]). By Theorem 2.4, X is finitistic. □

Theorem 2.6. *Suppose X is a separable metric space and A is a finitistic subset of X. There is a G_δ-subset B of X containing A so that B is finitistic.*

Proof. Let Z be a compact subset of A so that $A - U$ is finite-dimensional for every open neighborhood U of Z in A. Let U_n, $n \geqslant 1$, be a basis of decreasing neighborhoods of Z in X. Since $A - U_n$ is finite-dimensional, there exists a decreasing sequence $V_{n,m}$ of open neighborhoods of $A - U_n$ in X such that $\bigcap_{m \geqslant 1} V_{n,m}$ is finite-dimensional (see [20, 1.5.11, p. 34]). Define B as $\bigcap_{m,n}(V_{n,m} \cup U_n)$. Notice that $B - U_n \subset \bigcap_m (V_{n,m} - U_n)$ is finite-dimensional. By Theorem 2.4, B is finitistic. □

3. Maps of finitistic spaces

In this section we generalize results of classical dimension theory on maps which raise or lower covering dimension. The following two fundamental theorems can be found in [20, pp. 196–200]:

Theorem 3.1 (On Dimension-Raising Mappings for dim). *Suppose $f: X \to Y$ is a closed, surjective map of normal spaces such that there is an integer k with $f^{-1}(y)$ containing at most k elements for all $y \in Y$. Then,* $\dim(Y) \leqslant \dim(X) + k - 1$.

Theorem 3.2 (On Dimension-Lowering Mappings for dim). *Suppose $f: X \to Y$ is a closed, surjective map of a normal space X onto a weakly paracompact normal space Y such that there is an integer k with* $\dim(f^{-1}(y)) \leqslant k$ *for all $y \in Y$. Then,* $\dim(X) \leqslant \dim(Y) + k$.

First, we generalize a well-known fact that if $f: X \to Y$ is perfect and Y is compact, then X is compact:

Theorem 3.3. *Suppose $f: X \to Y$ is a closed map of paracompact spaces such that $f^{-1}(y)$ is finitistic for all $y \in Y$. If Y is compact, then X is finitistic.*

Proof. Suppose $g: X \to K$ is a map to a metric simplicial complex K. Consider the set $\mathcal{F}$ of all closed subsets A of Y such that $g|f^{-1}(A)$ has a finite-dimensional K-approximation. One of the assumptions of this theorem is that $\{y\} \in \mathcal{F}$ for all $y \in Y$. Using Lemma 1.8 one gets that each $y \in Y$ has a closed neighborhood $A_y \in \mathcal{F}$. Since Y is compact, Corollary 1.9 implies that $Y \in \mathcal{F}$ which completes the proof. □

One cannot weaken the assumptions of Theorem 3.3 by assuming that Y is finitistic rather than compact. Indeed, let Q be Hilbert cube. The projection $f: Q \times \mathbb{Z} \to \mathbb{Z}$ has compact fibers, $\mathbb{Z}$ being discrete is finitistic, and $Q \times \mathbb{Z}$ is not finitistic (use Theorem 2.4 or 0.6). However, we may put additional restriction on the fibers of f as seen in the next result:

Theorem 3.4. *Suppose $f: X \to Y$ is a closed map of paracompact spaces such that $f^{-1}(A)$ is finite-dimensional for all finite-dimensional closed subsets A of Y. If B is a closed, finitistic subset of Y, then $f^{-1}(B)$ is finitistic.*

Proof. Let Z be a compact subset of B such that $B - U$ is finite-dimensional for every open neighborhood U of Z in B (see Theorem 2.4). By Theorem 3.3, $f^{-1}(Z)$ is finitistic. Suppose V is an open neighborhood of $f^{-1}(Z)$ in $f^{-1}(B)$. On can find an open neighborhood U of Z in B with $f^{-1}(U) \subset V$. Since $B - U$ is finite-dimensional, so is $f^{-1}(B - U)$. Now, $f^{-1}(B) - V$ is a closed subset of $f^{-1}(B - U)$ and, therefore, is finite-dimensional. By Corollary 2.3, $f^{-1}(B)$ is finitistic. □

Here is a generalization of Theorem 3.2:

Corollary 3.5. *Suppose $f : X \to Y$ is a closed map of paracompact spaces such that there is $k \geqslant 0$ with $\dim(f^{-1}(y)) \leqslant k$ for all $y \in Y$. If Y is finitistic, then so is X.*

Proof. Theorem 3.2 says that $f^{-1}(A)$ is finite-dimensional for all finite-dimensional closed subsets A of Y. Use Theorem 3.4. □

As an application we get a generalization of a result of Deo and Singh [4] (see Theorem 0.3) regarding compact Lie group actions. The generalization applies to all compact group actions:

Corollary 3.6. *Suppose a compact, finite-dimensional group G acts on a paracompact space X. If X/G is finitistic, then so is X.*

Proof. It is well known that the projection $\pi : X \to X/G$ is closed. Indeed, if A is closed in X, then $C = \{(b, g) \in X \times G \mid b \cdot g \in A\}$ is closed in $X \times G$ as the multiplication is continuous. Since the projection $p : X \times G \to X$ is closed, $p(C)$ is closed. Notice that $p(C) = \pi^{-1}(\pi(A))$ which implies that $\pi(A)$ is closed in X/G. Also, X/G is paracompact as paracompactness is preserved by closed mappings. Use Corollary 3.5. □

Notice that the converse implication (i.e., if X is finitistic, then X/G is finitistic) known to be true for Lie groups G (see [6] and Theorem 0.2) does not hold for general compact and finite-dimensional groups G. A counterexample to that implication can be easily constructed using the following result of Dranishnikov and West:

Theorem 3.7 [10]. *Let $G = \prod_{i=1}^{\infty}(\mathbb{Z}/p)_i$ be the infinite product of copies of $\mathbb{Z}/p$, each copy being given the discrete topology. For each integer $n \geqslant 3$ and each prime p there is an action of G on a compact, two-dimensional metric space X_n such that $\dim(X_n/G) = n$.*

Indeed, $X = \bigoplus_{n=1}^{\infty} X_n$ (the discrete union of all X_n) is two-dimensional and admits an action of G so that X/G is not finitistic.

Let us show how to reduce Deo and Tripathi result (see Theorem 0.2) to the finite-dimensional case:

Theorem 3.8. *Suppose G is a compact topological group. The following conditions are equivalent*:

(a) *for any action of G on a finite-dimensional, paracompact space X, the orbit space X/G is finite-dimensional,*
(b) *for any action of G on a finitistic, paracompact space X, the orbit space X/G is finitistic.*

Proof. (a) $\Rightarrow$ (b) Suppose G acts on a finitistic, paracompact space X. Let Z be a compact subset of X so that $X - U$ is finite-dimensional for every neighborhood U of Z in X

(see Theorem 2.4). Let $\pi : X \to X/G$ be the projection. Notice that $\pi(Z)$ is compact. If V is an open neighborhood of $\pi(Z)$ in X/G, then $(X - \pi^{-1}(V))/G = (X/G) - V$ is finite-dimensional. Hence, by Theorem 2.4, X/G is finitistic.

(b) $\Rightarrow$ (a) Suppose there is a finite-dimensional, paracompact space X so that X/G is infinite-dimensional for some action of G on X. Notice that G acts on $X \times \mathbb{Z}$ with $(X \times \mathbb{Z})/G = (X/G) \times \mathbb{Z}$ being non-finitistic. This contradicts (b) as $X \times \mathbb{Z}$ is finitistic. □

Since compact Lie groups are known to satisfy condition (a) of 3.8, Theorem 0.2 of Deo and Tripathi [6] follows.

Theorem 3.9. *Suppose $f : X \to Y$ is a closed map of paracompact spaces such that $f(A)$ is finite-dimensional for all finite-dimensional closed subsets A of X. If B is a closed, finitistic subset of X, then $f(B)$ is finitistic.*

Proof. Let Z be a compact subset of B so that $B - U$ is finite-dimensional for all open neighborhoods U of Z in B (see Theorem 2.4). Notice that $f(Z)$ is compact. Suppose V is an open neighborhood of $f(Z)$ in $f(B)$. Since $B - f^{-1}(V)$ is finite-dimensional, so is $f(B - f^{-1}(V)) = f(B) - V$. By Theorem 2.4, $f(B)$ is finitistic. □

Here is a generalization of Theorem 3.1:

Corollary 3.10. *Suppose $f : X \to Y$ is a closed, surjective map of paracompact spaces such that there is an integer k with $f^{-1}(y)$ containing at most k elements for all $y \in Y$. If X is finitistic, then so is Y.*

Proof. Theorem 3.1 says that $f(A)$ is finite-dimensional for all finite-dimensional closed subsets A of X. Use Theorem 3.9. □

4. Finitistic spaces and cohomological dimension

The purpose of this section is to apply our characterizations of paracompact, finitistic spaces and present a simple proof of part (a) of Theorem 0.4. Here is a generalization of part (a) of Theorem 0.4:

Theorem 4.1. *Suppose X is a paracompact, finitistic space and K is a metric simplicial complex which is homotopy equivalent to a CW complex L whose skeleta are finite. If K is an absolute extensor of X, then K is an absolute extensor of the Čech–Stone compactification βX of X. If K is an absolute extensor of βX, and is complete, then K is an absolute extensor of X.*

Proof. Notice that any map $f : Y \to K$ from a paracompact, finitistic space Y to K factors up to homotopy through a compact polyhedron. Indeed, let $h : K \to L$ be a homotopy equivalence, where L is a CW complex whose skeleta are finite. We may assume that

$h(K^{(n)}) \subset L^{(n)}$ for all n. Choose a homotopy inverse $g : L \to K$ of h so that $g(L^{(n)}) \subset K^{(n)}$ for all n. Now, there is an m-dimensional K-approximation f' of f for some m (see Theorem 2.1), and $h \circ f' : Y \to L^{(m)}$ composed with $g|L^{(m)}$ is homotopic to f.

Suppose K is an absolute extensor of X. It suffices to show that any map $f : A \to K$, A closed in βX, extends over βX up to homotopy. Extend f to $g : B \to K$, where B is a closed neighborhood of A in βX. Since K is an absolute extensor of X, there is an extension $h : X \to K$ of $g|B \cap X$. Notice that h and f can be pasted together to give a map $F : A \cup X \to K$ extending f. Since $A \cup X$ is finitistic and paracompact (see Corollary 2.5(a)), F has an m-dimensional K-approximation $G : A \cup X \to K$ for some m (see Theorem 2.1). Now, G can be factored up to homotopy through a compact polyhedron. Since every map from X to a compact polyhedron has a unique extension over βX, G extends up to homotopy over βX.

Suppose K is complete and is an absolute extensor of βX. Again, as K is an absolute neighborhood extensor of all paracompact spaces (see [21, p. 109]), it suffices to show that any map $f : A \to K$, A closed in X, extends over X up to homotopy. Since A is finitistic, f has an m-dimensional K-approximation $F : A \to K$ for some m (see Theorem 2.1). Now, F can be factored up to homotopy through a compact polyhedron. Since every map from A to a compact polyhedron has a unique extension over βA, F extends up to homotopy over βA. Subsequently, as K is an absolute extensor of βX, this extension can be further extended over βX. Restrict the resulting map to X in order to obtain a homotopy extension of f. $\square$

Notice that Theorem 4.1 is a generalization of Theorem 0.4. Indeed, if G is a finitely generated Abelian group, then for each n there is an Eilenberg–MacLane complex $K(G, n)$ whose skeleta are finite and which is homotopy equivalent to a locally finite (and hence complete), countable metric simplicial complex. As can be found in [23] (see also [11] or [3] for locally compact spaces), $\dim_G(X) \leqslant n$ means that every map $f : A \to K(G, n)$, A closed in X, extends over X up to homotopy. If $K(G, n)$ is an absolute neighborhood extensor of $X \times I$, then one can use a Homotopy Extension Theorem and obtain a precise extension of f.

Added in proof. Jack Segal pointed out to the authors that Theorem 0.5 has been proved independently by Y. Hattori (see “A note on finitistic spaces” in Questions Answers Gen. Topology 3 (1) (1985) 47–55).

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